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STEADY LAMINAR FREE CONVECTION FROM INCLINED, ARBITRARILY SHAPED PLANE SURFACES

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Abstract—The note considers steady laminar boundary layer free convection from plane arbitrarily shaped surfaces for quasi-two-dimensional conditions. An attempt is made to represent the effect of surface geometry using a concept of "geometrical effectiveness". Illustrative results are obtained for plane surfaces and for bodies having a surface composed of plane sides.

NOMENCLATURE

- x, distance upwards from leading edge;
- y, distance normally outwards from surface;
- z, distance across surface;
- X, length of surface in direction of flow;
- Z, width of surface;
- A, arbitrary function of z;
- f, component of "g" in direction of flow;
- β, cubical coefficient of expansion;
- ν, kinematic viscosity;
- κ, thermal conductivity;
- q, heat flux density;
- T, ϕ, temperature;
- Nu, Nusselt number;
- G, Grashof number;
- Pr, Prandtl number;
- Ra, Rayleigh number;
- ψ, f, stream function;
- η, similarity variable.

Subscripts

- 0, surface;
- ∞, infinity;
- x, X length.

INTRODUCTION

It is easily shown that if lateral fluid velocities are sufficiently small the equations governing steady three-dimensional laminar boundary layer free convection from a plane surface reduce to their two-dimensional form, thus permitting a quasi-two-dimensional treatment of certain problems [1]. This note is an application of the quasi-two-dimensional treatment to heated plane surfaces of arbitrary shape.

Because of the nature of the treatment, we have at our disposal all previous two-dimensional solutions but for simplicity we will restrict ourselves to similarity flows [2–4]. For any particular similarity solution, the general heat-transfer relation for a strip of unit width in the z direction (see Fig. 1) may be written as,

$$\overline{Nu_x} = c\, Ra_x^{1/4} \tag{1}$$

where the coefficient c depends upon the Prandtl number and the form of the variation of surface temperature, which is given, in many important circumstances, by,

$$(T_0 - T_\infty)_x = A(z)\, x^n \text{ (see Appendix).}$$

The average heat-transfer rate for a strip of unit width is easily found from (1) as,

$$\bar{q}_0 = ck\left(\frac{\beta f}{\nu\kappa}\right)^{1/4}\left[\frac{(T_0 - T_\infty)_X^5}{X}\right]^{1/4}.$$

Hence the average heat-flux density for the entire arbitrary surface is given by

$$(\bar{q}_0)_{AV} = \int_0^Z \bar{q}_0\, X\, \mathrm{d}z/\text{plate area}$$

$$= ck\left(\frac{\beta f}{\nu\kappa}\right)^{1/4} \frac{\int_0^Z X^{3/4}\,[A(z)\, X^n]^{5/4}\,\mathrm{d}z}{\int_0^Z X\,\mathrm{d}z} \tag{2}$$

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which, it should be noted, depends upon the surface geometry indirectly only: that is, the determining factor is the shape of the curve of surface length (X) plotted against distance across the surface. Figure 1 shows the actual (broken line) and equivalent (X versus z) forms of an arbitrarily shaped surface.

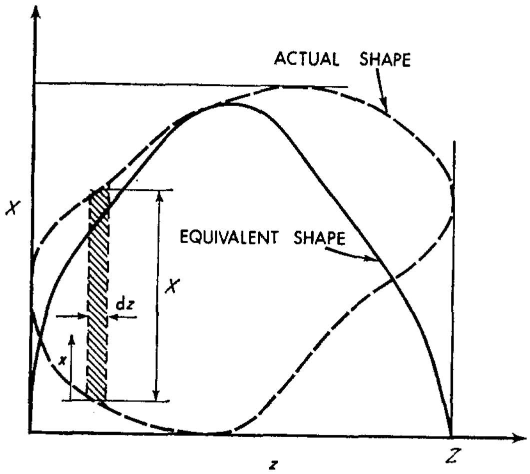

FIG. 1. Actual and equivalent shapes.

RELATIVE GEOMETRICAL EFFECTIVENESS

For a square surface (side L) having its lower edge perpendicular to the generating body force and having the same inclination (i.e., the same value of f) as the arbitrary surface, equation (2) becomes,

$$(\bar{q}_0)_{AV} = ck \left(\frac{\beta f}{\nu\kappa}\right)^{1/4} \frac{L^{3/4+5n/4} \int_0^L A(z)^{5/4}\,dz}{\int_0^L X\,dz}.$$

We may now define the "geometrical effectiv ness", ϵ_g, as the average heat-flux density of the arbitrarily shaped surface expressed as a fraction of the average heat flux density of a square of the same area. That is, we have,

$$\epsilon_g = \frac{\int_0^Z X^{3/4}[A(z)\,X^n]^{5/4}\,dz}{L^{3/4+5n/4}\int_0^L A(z)^{5/4}\,dz} \tag{3}$$

which also depends upon the equivalent rather than the actual surface shape.

There are actually two factors which govern the geometrical effectiveness; the surface shape and the surface orientation or "skew": skew is preferred to "inclination" which already has an accepted meaning, i.e. a surface not lying parallel to the generating body force.

SOME EXAMPLES

(a) *Vertical plates*

Obviously, the possible number of combinations of geometries, orientations and temperature distributions is enormous. Only two illustrative examples will be given here—the circle and the rectangle, both at uniform temperature. Before giving these, however, it is instructive to determine the form of ϵ_g if $A(z)$ is a constant and $n = 1/5$. From (3) we find that ϵ_g does not depend upon geometry at all, a fact which could easily have been anticipated since this condition corresponds to a uniformly heated plate, for which the geometry is unimportant.

For an isothermal circular plate (radius R) we have

$$\epsilon_g = \frac{2\int_0^R 2^{3/4}(R^2 - z^2)^{3/8}\,dz}{\pi^{7/8}R^{7/4}} = 1{\cdot}03,$$

which, as expected, is not a function of skew.

For an isothermal rectangle of the proportions shown in Fig. 2, the variation of ϵ_g with skew angle θ is given by

$$\epsilon_g = \left(\frac{a}{b}\right)^{1/8} \frac{[b/7a\,.\sin\theta + \cos\theta]}{\cos^{3/4}\theta},$$

$$\left[0 \leqslant \theta \leqslant \frac{\pi}{2} - \arctan\frac{b}{a}\right]$$

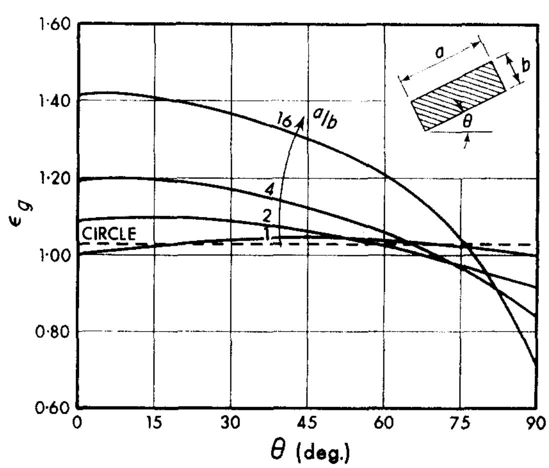

FIG. 2. Geometrical effectiveness of a skewed isothermal rectangle.

and

$$\epsilon_g = \left(\frac{b}{a}\right)^{1/8} \frac{[a/7b \, . \cos\theta + \sin\theta]}{\sin^{3/4}\theta}$$

$$\left[\frac{\pi}{2} - \arctan\frac{b}{a} \leqslant \theta \leqslant \frac{\pi}{2}\right].$$

This is shown plotted in the figure.

(b) *Solids*

Many solids have surfaces which are formed by plane elements. Some of these are amenable to the simple treatment of this note. If the inclination (to the direction of the generating body force) of each surface element is small, and the flow over one element does not substantially affect flow over the next, then the treatment given here should apply with reasonable accuracy. As an example, consider the tapering surface (only) of the m-sided pyramid (base side a_m) shown in Fig. 3. Rearranging (2), for an isothermal surface, as,

$$\frac{(\bar{q}_0)_{AV}}{ck\left(\frac{\beta f}{\nu\kappa}\right)^{1/4}} = \frac{\int_0^Z X^{3/4}\,dz}{\int_0^Z X\,dz}$$

we see that the right-hand side is a property of geometry alone. Considering pyramids of equal volume and equal base area (taken equal to a_4^2 for convenience) we find that,

$$\frac{7\,(\bar{q}_0)_{AV}\, a_4^{1/4}}{8\,ck\left(\frac{\beta f}{\nu\kappa}\right)^{1/4}} = \left[\frac{\cot\pi/m}{m} + \left(\frac{b}{a_4}\right)^2\right]^{-1/8}$$

which is shown plotted in Fig. 3. It is interesting to note that as $m \to \infty$,

$$\frac{\cot\pi/m}{m} \to \frac{1}{\pi}$$

and hence we conclude that,

$$(\overline{Nu_x})_{\text{cone}} = \tfrac{8}{7}\,(\overline{Nu_x})_{\text{flat plate}}$$

x being measured along the surface in the direction of flow in each case. The figure indicates that except for very shallow pyramids the heat transferred is almost independent of m.

CONCLUSIONS

Using a simple quasi-two-dimensional treatment it has been shown that the heat transferred in laminar free convection from an arbitrarily shaped plane surface depends upon an equivalent rather than the actual surface shape. The concept of a "geometrical effectiveness" enables a comparison to be made between the arbitrary surface and a square of the same area.

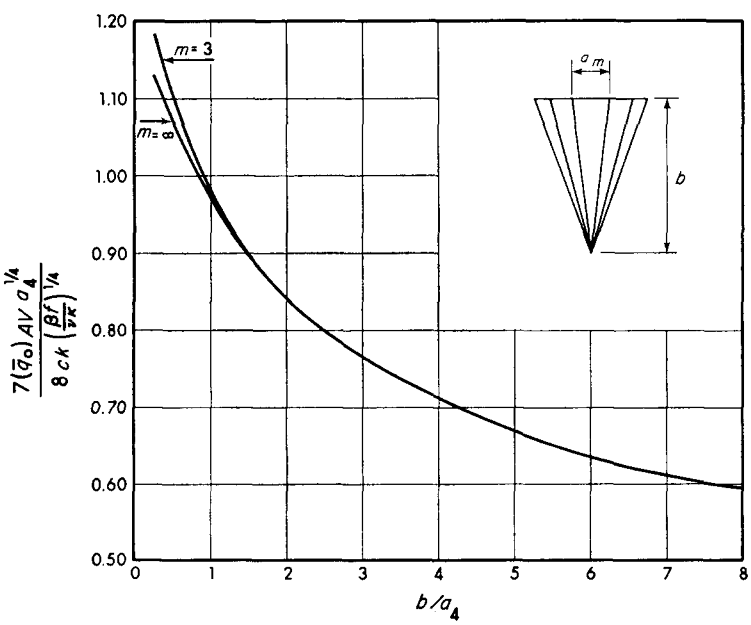

FIG. 3. Average heat transfer from an m-sided pyramid.

The analysis may be extended to solid bodies if the body surface is composed of plane elements arranged such that the flow over each element is independent of the flow over the rest.

ACKNOWLEDGEMENT

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APPENDIX

Permissible Longitudinal Surface Temperature Distributions

Seeking two-dimensional similarity solutions we may put

$$\psi(x, y) = f(\eta)\,\xi(x)$$

$$\eta = y\,\zeta(x)$$

$$\phi = \frac{G(x, y)}{G_0(x)}.$$

Substituting these into the boundary-layer forms of the governing equations we obtain [4]

$$f''' + \alpha_1 f'' f - (\alpha_2 + \alpha_1)(f')^2 + \alpha_3 \phi = 0$$

$$\frac{1}{Pr}\phi'' + \alpha_1 f\phi' - \alpha_4 f'\phi = 0$$

where the coefficients α are constants defined by

$$\alpha_1 = \frac{1}{\zeta}\frac{d\xi}{dx} \tag{4a}$$

$$\alpha_2 = \frac{\xi}{\zeta^2}\frac{d\zeta}{dx} \tag{4b}$$

$$\alpha_3 = \frac{G_0}{\xi\zeta^3} \tag{4c}$$

$$\alpha_4 = \frac{\xi}{G_0\zeta}\frac{dG_0}{dx}. \tag{4d}$$

Equations (4a) and (4c) taken together give

$$\tfrac{3}{4}\,\xi^{4/3} = \alpha_1 \int \left(\frac{G_0}{\alpha_3}\right)^{1/3} dx + \text{const.}$$

or, putting the arbitrary constant equal to zero,

$$\xi = \left(\frac{4J}{3}\right)^{3/4} \tag{5}$$

where

$$J(x) = \alpha_1 \int \left(\frac{G_0}{\alpha_3}\right)^{1/3} dx.$$

Equation (4c) may now be written as

$$\zeta = \frac{1}{\alpha_1}\left(\frac{3}{4J}\right)^{1/4}\frac{dJ}{dx}$$

which, when combined with (5) and (4b) gives

$$\frac{\gamma - 1}{\gamma}\left(\frac{dJ}{dx}\right)^2 = J\frac{d^2J}{dx^2} \tag{6}$$

where

$$\gamma = \frac{4\alpha_1}{3(\alpha_1 - \alpha_2)}.$$

Permissible solutions to (6) are

$$\gamma - 1 \neq \gamma; \quad J = (\lambda_1 x + \lambda_2)^{\gamma} \tag{7a}$$

$$\gamma - 1 = \gamma; \quad J = \omega\, e^{\epsilon x} \tag{7b}$$

where λ_1, λ_2, ω and ϵ are all arbitrary.

The permissible forms of G_0 which result from (7) are the same as those given in [4], with one important distinction. From (7) it can be seen that the exponential distribution is not really a separate form, but merely the asymptote of (7a) as $\gamma \to \infty$. In general then, the permissible surface temperature distribution is given by

$$G_0(x) \propto (\lambda_1 x + \lambda_2)^{3(\gamma-1)}.$$

If the leading edge takes the temperature of the fluid at infinity, then $G_0(0) = 0$ and hence $\lambda_2 = 0$, i.e.

$$G_0(x) \propto x^{3(\gamma-1)}$$

which is also a valid form if the surface is isothermal, i.e. $\gamma = 1$.

Résumé—La note considère la convection libre stationnaire avec une couche limite laminaire à partir de surfaces planes de forme arbitraire pour des conditions presque bidimensionnelles. On fait une hypothèse pour représenter l'effet de la géométrie de la surface en utilisant un concept d' "efficacité géométrique".

On obtient des résultats explicatifs pour des surfaces planes et pour des corps ayant une surface composée de faces planes.

Zusammenfassung—Es wird die stationäre freie Konvektion mit laminarer Grenzschicht an ebenen Flächen beliebiger Gestalt für quasi zweidimensionale Verhältnisse betrachtet. Um den Einfluss der Oberflächengeometrie wiederzugeben, wurde ein "geometrischer Wirkungsgrad" angenommen. Anschauliche Ergebnisse liessen sich für ebene Flächen und für Körper, deren Oberflächen eben sind, erhalten.

Аннотация—Рассматривается свободная конвекция в стационарном квазидвухмерном ламинарном пограничном слое для плоских поверхностей произвольной формы. Сделана попытка представить эффект геометрии поверхности с помощью использования понятия «геометрической эффективности». Получены наглядные результаты для плоских поверхностей и для тел, поверхность которых составлена из плоских сторон.